Note that using the statements of Sects. 2 and 3 enables us to extend the results of Theorems 2 and 3 to the case when the characteristic numbers of the matrix A_{22} in (1.1) have real parts that change signs at one point of the interval (t_0, T).

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ON THE DEFINITION OF VARIATIONS IN THE MECHANICS OF CONTINUOUS MEDIA*

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The basic forms of variations used in the mechanics of continuous media are presented, and relations between various types of variations of vectors and tensors are established.

The construction of new more complex models of continuous media can be based on the use of the variational equation /1/. In constructing models of continuous dislocations of plastic and solid media interacting with an electromagnetic field (in Newtonian mechanics as well as in the theory of relativity) /2-6/ and, also, a number of other models, it is necessary to deal with variations of various types of different quantities, such as scalars, vectors, and tensors which can be considered as functions of Euler or Lagrangian coordinates. Hence it is necessary to have established connections between various types of variations which are of the same nature as the variable functions.

Below we consider some of the simplest types of variations used to construct models of solid media in the special theory of relativity. We shall denote by x^i $(i = 1, 2, 3, 4)$ the Euler coordinates and by ξ^a $(a = 1, 2, 3, 4)$ the Lagrangian coordinates of four-dimensional Minkowski space, assuming that the global, coordinates x^4 and ξ^4 have a temporal nature $x^4 = ct$, $\xi^4 = c\tau$, (c is the velocity of light in a vacuum).

In the coordinate system x^i with basis vectors э_i defined as unit vectors tangent to the lines $x^i = \mathrm{const}$, and the particle world lines determined by the equations $x^i = x^i(\xi^a)$ (the law of motion of a point with Lagrangian coordinates relative to system x^i). Here and henceforth Greek indices run through the numbers 1,2,3, and the lower case Latin letters through 1,2,3,4.

At each point of the Minkowski four-dimensional space-time we may introduce covariant and contravariant basis vectors $(\text{э}_i,$ and $\text{э}^i, \text{э}_a{}^\wedge$ and $\text{э}^{\wedge a}$ for coordinates x^i and for systems ξ^α, respectively, connected by the equations

$$\text{э}_a{}^\wedge = \frac{\partial x^i}{\partial \xi^a}\text{э}_i = x_a{}^i\text{э}_i, \quad \text{э}_i = \frac{\partial \xi^a}{\partial x^i} = \xi_i{}^a\text{э}_a{}^\wedge$$

When constructing models of media and fields besides the law of motion one has to consider various scalar, vector, and tensor fields that represent mechanical, physical, or chemical characteristic of the phenomena and processes investigated which are functions of the coordinates x^i or ξ^a (for details of these characteristics see, e.g., /6/). In problems related to specifying or determining the laws of motion of the solid medium, and the laws of variation

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of fields, except real laws and fields $\mu^{(A)}$ $(A = 1, 2, \ldots)$, we can mentally introduce the varied law of motion and varying scalar, vector, and tensor fields $\mu^{(A)'}$. The difference between the varied value $\mu^{(A)'}$ and real value $\mu^{(A)}$ is usually called the variations of the function $\mu^{(A)}$. The variation of the functions may be introduced by various means in conformity with various definitions. For instance, the variation of the function $\mu^{(A)}$ at the point M can be defined as the increment of that function due to the displacement from a given point M to close to point M', or as an increase due to infinitely small transformation of the coordinates at the fixed point M, etc. When determining the variations of scalars, vectors and tensors need not necessarily require that the variations represent increments of functions relative to some ancillary parameters which determine the global fields of varied functions. Note that it is advisable when varying tensors to determine their variations so that they represent tensors of the same rank with the same structure of indices as that of the varied tensor.

Below, within the limits of the four-dimensional space only infinitely small variations of scalars, vectors, and tensors are considered, and the simplest connections between various possible forms of variations of these quantities are established.

1. Variation of the law of motion. We shall define the variation of the law of motion assuming that the observer's system of coordinates x^i and the accompanying Lagrangian system of coordinates ξ^a are fixed. The variation of the law of motion of point M_0 which has fixed Lagrangian space coordinates ξ_0^α will be determined in the observer's system of coordinates by the equation

$$\delta x^i = x^{i'}(\xi_0^\alpha, \xi^4) - x^i(\xi_0^\alpha, \xi^4) \tag{1.1}$$

where $x^i(\xi_0^\alpha, \xi^4)$ is the world line of point M_0 in the observer's system of coordinates and $x^{i'}(\xi_0^\alpha, \xi^4)$ is the world line, close to the world line $x^i(\xi_0^\alpha, \xi^4)$ that corresponds to the varied law of motion of point M_0. For a fixed ξ^4 variation δx^i represents a small possible displacement of point M_0 with coordinates x^i to a near point with coordinates $x^{i'}$. With this definition of the variation of the law of motion, the real displacements dx^i of point $M_0(\xi_0^\alpha)$ are among the possible ones; the variations δx^i become real displacements, if

$$\delta x^i = \frac{\partial x^i}{\partial \xi^4} d\xi^4$$

(Here and henceforth the index zero on the Lagrangian coordinates of point M is omitted, we denote its coordinates by ξ^a assuming at the same time that M is an arbitrary, but fixed point of the solid medium).

In the observer's fixed system of coordinates x^i the connection between the basis vectors of that system at points x^k and $x^k + \delta x^k$ is defined by the equations

$$\mathfrak{d}_i(x^k + \delta x^k) = \mathfrak{d}_i(x^k) + \delta x^s \Gamma_{is}^l \mathfrak{d}_l(x^k)$$

where Γ_{is}^l are Christoffel symbols. The basis vectors of the observer's system of coordinates and the accompanying system of coordinates for the actual and the varied law of motion are connected by the formulas

$$\mathfrak{d}_a\hat{} = x_a^i \mathfrak{d}_i(x^k) \tag{1.2}$$

$$\mathfrak{d}_a\hat{}' = \frac{\partial x^{i'}}{\partial \xi^a} \mathfrak{d}_i(x^k + \delta x^k) = \mathfrak{d}_a\hat{} + x_a^s \nabla_s \delta x^i \mathfrak{d}_i$$

where ∇_s is the operator of covariant differentiation in the observer's system of coordinates.

Generally the remainder $\mathfrak{d}_a\wedge' - \mathfrak{d}_a\wedge$ is the variation of the vectors of the Lagrangian frame of reference stipulated by the variation of the law of motion, which is expressed by the variation of motion in accordance with the formula

$$\mathfrak{d}_a\hat{}' - \mathfrak{d}_a\hat{} = \delta_x \mathfrak{d}_a\hat{} = x_a^s \nabla_s \delta x^i \mathfrak{d}_i \tag{1.3}$$

Since the variation of the law of motion is defined inthe observer's fixed system of coordinates, i.e. it is assumed by the definition that $\delta_x \mathfrak{d}_i = 0$, from formula (1.3) there follows the expression for the variation of the elements of the transformation matrix $\| x_a^i \|$

$$\delta_x x_a^i = x_a^s \nabla_s \delta x^i \tag{1.4}$$

From the equations $\xi_k^a x_a^i = \delta_k^i$ and $\xi_i^b x_a^i = \delta_a^b$ we obtain the expressions for the variations $\delta_x \mathfrak{d}\wedge^a$, $\delta_x g_{ab}\wedge$ and $\delta_x g\wedge^{ab}$

$$\delta_x \mathfrak{d}\hat{}^a = -\xi_i^a \nabla_k \delta x^i \mathfrak{d}^k \tag{1.5}$$

$$\delta_x g\hat{}_{ab} = g_{ij}(x_b^i x_a^s + x_a^j x_b^s) \nabla_s \delta x^i$$

$$\delta_x g\hat{}^{ab} = -g^{ij}(\xi_j^b \xi_s^a + \xi_j^a \xi_s^b) \nabla_i \delta x^s$$

2. Variations of the vectors and tensors.

When designing models of solid media and fields, alongside the parameters defining the simulated physical events in terms of simplest scalars and vectors dynamic and kinematic characteristics (such as density, velocity, etc.) some supplementary scalar, vector and tensor parameters may also appear in the equations. These parameters may be conditionally divided in two classes, namely, the parameters $\mu^{(A)}$ ($A = 1, 2, \ldots$), which define the physical state of the solid medium or field (for instance, entropy, the electric and magnetic field strength, antisymmetric tensors that define the inner angular momentum, etc.), and the parameters $\mathbf{K}^{(B)}$ ($B = 1, 2, \ldots$), that define the geometrical and physical properties of the solid medium (for instance, the permittivity and permeability of the medium, the elastic moduli, tensors defining the anisotropic properties of the medium etc.). Parameters which define the physical state of the medium can change independently of the geometric, kinematic, and dynamic properties of the medium and may be used to describe the interaction between the medium and external heat flow, with the electromagnetic field, etc. Parameters of the second kind can be universal physical constants or depend on the coordinates x^i of ξ^a. These parameters among the arguments of thermodynamic functions separate the specific solid medium from the multiplicity of all kinds of possible media. The different meaning of these two forms of parameters compels us to treat their variation differently, as well as the equations obtained when varying them.

Note that the question of a parameter belonging to this or that kind of type must be resolved in each specific case. For instance, when setting up models in the context of Newtonian mechanics, the components of the metric tensor g_{ij} and the properties (Euclidean) of space have to be related to the physically varied constants or to known functions of the coordinates, while when setting up models in the general theory of relativity, the components of the metric tensor are the unknown functions and are varied.

Variations of an arbitrary vector or tensor field $\mu^{(A)}$ in Newtonian mecahnics and in the special theory of relativity can be introduced onthe assumption that the observer's system of coordinates x^i as well as the Lagrangian system of coordinates ξ^a are fixed. For simplicity, we assume that $\mu^{(A)}$ is a vector

$$\boldsymbol{\mu} = \mu^i(x^k)\,\mathbf{э}_i(x^k) = \mu^{\wedge a}(\xi^b)\,\mathbf{э}_a{}^{\wedge}(\xi^b) \tag{2.1}$$

Let μ' be an arbitrary vector field which differs insignificantly from the vector field $\boldsymbol{\mu}$, which has the components $\mu^{i'}$ and $\mu^{\wedge a'}$ in the same frame of reference $\mathbf{э}_i$ and $\mathbf{э}_a{}^{\wedge}$ at the point M. The partial variation $\partial\mu$ of the vector field $\boldsymbol{\mu}$ is defined at point M as the remainder

$$\partial\boldsymbol{\mu} = \boldsymbol{\mu}' - \boldsymbol{\mu} \tag{2.2}$$

By this definition the partial variations of the components of the vector $\boldsymbol{\mu}$ are also components of the vector and for them the formulae

$$\begin{gathered}\partial\boldsymbol{\mu} = \partial\mu^i\mathbf{э}_i = \partial\mu_i\mathbf{э}^i = \partial\mu^{\wedge a}\mathbf{э}_a{}^{\wedge} = \partial\mu_a{}^{\wedge}\mathbf{э}^{\wedge a} \\ \partial\mu^i = x_a{}^i\partial\mu^{\wedge a}, \quad \partial\mu^i = g^{ij}\partial\mu_j, \quad \partial\mu_b{}^{\wedge} = g_{ab}^{\wedge}\partial\mu^{\wedge a}\end{gathered} \tag{2.3}$$

hold.

Defining the partial variations of the vector, we assume that the vector field $\boldsymbol{\mu}'$ (consequently, also variations of $\partial\mu$)) are arbitrary. From these in view of the supplementary assumptions we can separate partial variations of special form. For instance, it is possible to assume that partial variations of the components of the vector $\mu^{i'}$ are variations of vector components for infinitely small transformation of the observer's system of coordinates $\partial\mu^i = \delta_\eta\mu^i = \mu^s\nabla_s\,\delta\eta^i$ (see Sect. 3).

Partial variations of a tensor of any rank with an arbitrary structure of indices may be introduced by a formula similar to (2.2), when the partial variation of the tensor is a tensor of the same rank and the same structure of indices. The components of this tensor undergo transformation from the system of coordinates x^i to the system ξ^a using matrices $\| x_a{}^i \|$ and $\| \xi_i{}^a \|$, and in particular, since in both the Newtionian mechanics and the special theory of relativity the metric tensor $\mathbf{G} = g_{ij}\,\mathbf{э}^i\mathbf{э}^j = g_{ab}{}^{\wedge}\mathbf{э}^{\wedge a}\mathbf{э}^{\wedge b}$ is assumed to be given, hence $\partial g_{ij} = \partial g_{ab}{}^{\wedge} = 0$.

Since by the definition of a partial variation, the systems of coordinates x^i and ξ^a are by assumption fixed, the symbols of partial variation and the operator of covariant differentiation are interchangeable.

The variation of the vector field $\boldsymbol{\mu}$ at the point M in Lagrangian coordinates ξ^a due to the variation of the law of motion, can be defined as the remainder of vector field values at points $x^{k'}(\xi^a)$. and $x^k(\xi^a)$

$$\delta_x\boldsymbol{\mu} = \mathbf{u}(x^{k'}) - \boldsymbol{\mu}(x^k) \tag{2.4}$$

The right side of (2.4) can be written in the form

$$\delta_x\boldsymbol{\mu} = \delta_x\mu^i\mathbf{э}_i = \delta x^k\nabla_k\mu^i\mathbf{э}_i \tag{2.5}$$

where the variations of the contravariant and covariant components $\delta_x\mu_j$ and $\delta_x\mu^i$ are connected by the equation

$$\delta_x\mu^i = g^{ij}\delta_x\mu_j, \qquad \delta_x\mu_j = g_{ij}\delta_x\mu^i$$

i.e. the manipulation of indices is carried out on components of the metric tensor g_{ij}.

The variations of a tensor of arbitrary rank with an arbitrary structure of indices when the law of motion is varied can be calculated by formulae similar to (2.5). For example, the variation of a second rank tensor is

$$\delta_x \mathbf{T} = \delta x^k \nabla_k T_{.j}^{i\cdot} \mathfrak{э}_i \mathfrak{э}^j \quad (\mathbf{T} = T_{j.}^{\ i\cdot} \mathfrak{э}_i \mathfrak{э}^j)$$

The last formula shows that variations of the Euler components of the metric tensor, when the law of motion is varied, are equal to zero

$$\delta_x g_{ij} = \delta_x g^{ij} = 0 \tag{2.6}$$

In the general theory of relativity the components of the metric tensor of the observer's system of coordinates g_{ij} are the unknown functions that define the Riemannian space geometry, which are to be determined by solving specific problems and must be related to varied parameters of the type $\mu^{(A)}$. In designing models in the general theory of relativity partial variations of the metric tensor components g_{ij} are arbitrary and non-zero. Using the equation

$$\partial \frac{\partial g_{ij}}{\partial x^k} = \frac{\partial}{\partial x^k}\,\partial g_{ij}, \quad \partial \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2}{\partial x^k \partial x^l}\,\partial g_{ij}$$

for partial variations of the Christoffel symbols and the covariant derivative of the tensor $\mu^{(A)}$, we have /6/

$$\partial\Gamma_{ij}^{\ k} = \frac{1}{2}\left[g^{kn}(\delta_j^{\ s}\delta_i^{\ q} + \delta_i^{\ s}\delta_j^{\ q}) - g^{ks}\delta_i^{\ n}\delta_j^{\ q}\right]\nabla_s \partial g_{nq}$$

$$\partial\nabla_i\mu^B = \nabla_i\partial\mu^B + F_{Ck}^{Bj}\mu^C\partial\Gamma_{ij}^{\ k}$$

where B is the collective index of components of the tensor of arbitrary rank $\mu^{(A)}$; $F_{Cs}^{\ \ Bj}$ denotes the sum of products of Christoffel symbols of completely defined form. For example, for a second-rank tensor (when $B = mn$ and $C = lr$), we have

$$F_{Cs}^{Bj} = \delta_r^{\ n}\delta_s^{\ m}\delta_l^{\ j} + \delta_s^{\ n}\delta_l^{\ m}\delta_r^{\ j}$$

Taking as the basic assumption that the equation

$$\delta_x(\mu^i\mathfrak{э}_i) = \delta_x(\mu^{\wedge a}\mathfrak{э}_a^{\ \wedge}) \tag{2.7}$$

is satisfied when the law of motion is varied, i.e. the variation of the vector μ, due to the variation of the law of motion, is dependent on which coordinates (Eulerian of Lagrangian) are taken as the arguments of the varied vector. From (2.7) taking (1.4) and (2.5) into account we obtain the expressions for the variations of the Lagrangian components of the vector

$$\delta_x\mu^{\wedge a} = \xi_i^{\ a}\delta x^k\nabla_k\mu^i - \xi_i^{\ a}\mu^k\nabla_k\delta x^i \tag{2.8}$$

The variations of the Lagrangian components of a tensor of arbitrary rank with an arbitrary structure of the indices can be similarly determined. For example, the variations of the Lagrangian components to the second-rank tensor $\mathbf{T} = T_{.j}^{i\cdot}\mathfrak{э}_i\mathfrak{э}^j = T_{\cdot b}^{\wedge a\cdot}\mathfrak{э}_a^{\ \wedge}\mathfrak{э}^{\wedge b}$ are expressed in terms of variation of the law of motion by the formulae

$$\delta_x T^{\wedge a\cdot}_{\ \ \cdot b} = \xi_k^{\ a} x_b^{\ i}\delta x^s\nabla_s T_{\cdot i}^{k\cdot} - \xi_i^{\ a} T^{\wedge c\cdot}_{\ \ \cdot b}\nabla_c^{\ \wedge}\delta x^i + \xi_i^{\ c} T^{\wedge a\cdot}_{\ \ \cdot c}\nabla_b^{\ \wedge}\delta x^i \tag{2.9}$$

where $\nabla_c^{\ \wedge}$ is the operator of covariant differentiation in the Lagrangian system of coordiates. The formulae obtained enable us to determine the expression for the variation of $x_a^{\ i}$. If in the set of $x_a^{\ i}$ the upper indices are assumed fixed, then the set of $x_1^{\ i}$, $x_2^{\ i}$, $x_3^{\ i}$, $x_4^{\ i}$ can be considered as the Lagrangian components of the vector. It can be shown that the variations of these components are calculated by the formula

$$\delta_x x_a^{\ i} = \nabla_a^{\ \wedge}\delta x^i = x_a^{\ k}\nabla_k\delta x^i$$

which is the same as (1.4)

Variations of the Lagrangian components $\delta_x\mu^{\wedge a}$ and $\delta_x\, T_{.b}^{\wedge a\cdot}$, converted to the observer's system of coordinates x^i by the transformation formulae for the components of vectors and tensors, have the form

$$\begin{aligned} x_a^{\ i}\delta_x\mu^{\wedge a} &= \delta x^k\nabla_k\mu^i - \mu^k\nabla_k\delta x^i \\ x_a^{\ i}\xi_k^{\ b}\delta_x T^{\wedge a\cdot}_{\ \ \cdot b} &= \delta x^s\nabla_s T_{\cdot k}^{i\cdot} - T_{\cdot k}^{s\cdot}\nabla_s\delta x^i + T_{\cdot s}^{i\cdot}\nabla_k\delta x^s \end{aligned} \tag{2.10}$$

The sum of the partial variation of the Euler component of a vector (or tensor) and the variations of the Euler components of a vector (tensor) by varying the law of motion are complete variations of the vector (tensor) components, which for the second-rank vector and tensor have the form

$$\delta\mu^i = \partial\mu^i + \delta x^k \nabla_k \mu^i \qquad (2.11)$$
$$\delta T_{.j}^{i.} = \partial T_{.j}^{i.} + \delta x^k \nabla_k T_{.j}^{i.}$$

and the complete variation of Lagrangian components are calculated by the formulae

$$\delta_L \mu^{\wedge a} = \partial\mu^{\wedge a} + \delta_x \mu^{\wedge a} \qquad (2.12)$$

$$\delta_L T_{.b}^{\wedge a.} = \partial T_{.b}^{\wedge a.} + \delta_x T_{.b}^{\wedge a.}$$

According to these formulae it can be assumed that partial variations are determined for constant Euler coordinates, and complete variations are determined at constant Lagrangian coordinates (and the varied law of motion).

Taking into account that partial variations of the vector and tensor components are transformed when passing from the system of coordinates ξ^a to the system x^i by conventional transformation formulae, formula (2.12) may be written in the form

$$x_a{}^i \delta_L \mu^{\wedge a} = \partial\mu^i + \delta x^k \nabla_k \mu^i - \mu^k \nabla_k \delta x^i \qquad (2.13)$$
$$x_a{}^i \xi_k{}^b \delta_L T^{\wedge a}_{\cdot b} = \partial T^{i.}_{\cdot k} + \delta x^s \nabla_s T^{i.}_{\cdot k} - T^{s.}_{\cdot k} \nabla_s \delta x^i + T^{i.}_{\cdot s} \nabla_k \delta x^s$$

The expressions on the right side of (2.13) are conventionally denoted by $\delta_L \mu^i$ and $\delta_L T_{.k}^{i.}$, respectively, and are variations of the components of a vector and tensor of second rank, introduced in the Lagrangian system of coordinates and converted to the observer's system of coordinates (such variations are called absolute variations).

Formulae for variations of Lagrangian components of the metric tensor follow from (2.12) and have the form

$$\delta_L g_{\hat{a}\hat{b}} = \delta_x g_{\hat{a}\hat{b}} = \xi_i{}^c g_{\hat{c}\hat{b}} \nabla_a{}^\wedge \delta x^i + \xi_i{}^c g_{\hat{a}\hat{c}} \nabla_b{}^\wedge \delta x^i$$
$$\delta_L g^{\wedge ab} = \delta_x g^{\wedge ab} = -\xi_i{}^a g^{\wedge cb} \nabla_c{}^\wedge \delta x^i - \xi_i{}^b g^{\wedge ac} \nabla_c{}^\wedge \delta x^i$$

and the absolute variations of the Lagrangian components of the metric tensor transformed to the observer's system of coordinates have the form

$$\xi_i{}^a \xi_j{}^b \delta_L g_{\hat{a}\hat{b}} = g_{is} \nabla_j \delta x^s + g_{sj} \nabla_i \delta x^s$$
$$x_a{}^i x_b{}^j \delta_L g^{\wedge ab} = -g^{is} \nabla_s \delta x^j - g^{sj} \nabla_s \delta x^i$$

It follows from (2.11) and (2.13) that the complete variations of the components of the vector μ introduced relative to systems of coordinates x^i and ξ^a are no longer connected by the usual formulae of passing from one system of coordinates to the other

$$\delta\mu^i \neq x_a{}^i \delta_L \mu^{\wedge a}$$

This is due to the fact that, when the law of motion is varied, the basis vectors of the Lagrangian system of coordinates ξ^a are different for the varied and actual law of motion

By the definition of the tensor partial variation and the variation due to the law of motion, for real motions and processes, the partial variations vanish, while complete variations of the components of the tensors become actual increments of components in the system of coordinates x^i and ξ^a. For example, for the real increments of Euler components of a second-rank tensor we have

$$dT_{\cdot j}^{i\cdot} = u^k \nabla_k T_{\cdot j}^{i\cdot} d\xi^4$$

and for the increments of the Lagrangian components converted to the system of coordinates x^i

$$d_L T^{i.}_{.j} = u^k \nabla_k T^{i.}_{.j}\, d\xi^4 - T^{k.}_{.j} \nabla_k u^i\, d\xi^4 + T^{i.}_{.k} \nabla_j u^k\, d\xi^4$$

where u^i are components of the dimensionless vector of the 4-velocity.

Variations of scalar, vector and tensor parameters $\mathbf{K}^{(B)} = \mathbf{K} = {}^{(B)}(x^i) = \mathbf{K}^{(B)}(\xi^a)$ that define the geometric or physical properties of the solid medium, are defined by the formulae

$$\partial K^A = \partial K^{\wedge C} = 0$$
$$\delta K^A = \delta x^i \nabla_i K^A, \quad \delta_L K^{\wedge C} = K^A \delta_x \xi_A{}^C + \xi_A{}^C \delta x^i \nabla_i K^A$$

where A and C are the collective notation of the tensor indices $A = ijk\ldots$, $C = abc\ldots$, K^A and $K^{\wedge C}$ are the Eulerian and Lagrangian components of the tensor $\mathbf{K}^{(B)}$, and we denote by $\xi_A{}^C$ products of the form $\xi^i{}_a \xi_j{}^b \xi_k{}^c \ldots$

A more detailed analysis of various forms of variations of parameters $\mathbf{K}^{(B)}$ is given in /6/.

3. Variations of the vectors and tensor on transforming the observer's system. Let M_0 be a fixed point with Lagrangian coordinates $\xi_0{}^a$, $y_0{}^i$ and $x_0{}^i$ be its coordinates in the systems of coordinates y^i and x^i that correspond to two different frames of reference connected by the infinitely small transformation

$$y^i = x^i + \delta\eta^i(x^k) \tag{3.1}$$

Tetrads of the basis vectors $\mathfrak{s}_i(x^k)$ and $\mathfrak{s}_i'(y^k)$ of the system of coordinates x^i and y^i are connected with the tetrad of the Lagrangian system of coordinates at point M_0 by the equations

$$\mathfrak{s}^i(x^k) = x_a^{\ i}\mathfrak{s}^{\wedge a}(\xi_0^{\ b}),\quad \mathfrak{s}^{i'}(y^k) = y_a^{\ i}\mathfrak{s}^{\wedge a}(\xi_0^{\ b})$$

By an infinitely small transformation of coordinates (3.1) vectors $\mathfrak{s}^{i'}(y^k)$ convert into vectors $\mathfrak{s}^i(x^k)$ as the result of two successive operations performed on them, namely, parallel transfer from the point $x_0^{\ i} + \delta\eta^i(x_0^{\ k})$ to the point $x_0^{\ i}$, and the transformation of coordinates at the point $x_0^{\ i}$

$$\mathfrak{s}^{i'}(y^k) = \mathfrak{s}^i(x^k) + \nabla_s\delta\eta^i\mathfrak{s}^s(x^k)$$

The remainder

$$\delta_\eta\mathfrak{s}^i = \mathfrak{s}^{i'}(y^k) - \mathfrak{s}^i(x^k) = \nabla_s\delta\eta^i\mathfrak{s}^s(x^k) \tag{3.2}$$

represents the variation of the vectors of the observer's system of coordinates for an infinitely small transformation (3.1).

Variations of the elements of the matrix of transformation $x_a^{\ i}$ for transformation (3.1) may be introduced, using the equation $\delta_\eta\mathfrak{s}^i = \delta_\eta(x_a^{\ i}\mathfrak{s}^{\wedge a})$, taking into account that for transformation (3.1) of the coordinates, the vectors of the Lagrangian reference system remain unchanged

$$\delta_\eta x_a^{\ i} = x_a^{\ s}\nabla_s\delta\eta^i \tag{3.3}$$

The expression for variations of the vectors $\mathfrak{s}_k$ for transformation (3.1) can be obtained from the equations $\mathfrak{s}^i\mathfrak{s}_k = \delta_k^{\ i}$, and has the form

$$\delta_\eta\mathfrak{s}_k = -\nabla_k\delta\eta^i\mathfrak{s}_i \tag{3.4}$$

Since the tensor is invariant to transformation (3.1)

$$\delta_\eta(T^{A\cdot}_{\cdot B}\mathfrak{s}_A\mathfrak{s}^B) = 0$$

where A and B are the collective notation of the covariant tensor indices, and we denote by $\mathfrak{s}_A$ and $\mathfrak{s}^B$ the polyadic products $\mathfrak{s}_i\mathfrak{s}_j\mathfrak{s}_k\ldots$ and $\mathfrak{s}^m\mathfrak{s}^n\mathfrak{s}^s\ldots$, respectively. Hence on the basis of (3.2), from this equation

$$\mathfrak{s}_A\mathfrak{s}^B\delta_\eta T^{A\cdot}_{\cdot B} = -T^{A\cdot}_{\cdot B}\delta_\eta(\mathfrak{s}_A\mathfrak{s}^B) \tag{3.5}$$

we can obtain expressions for the variation of the tensor for transformation (3.1). For example, for variations of the components of the metric tensor g^{ij} and g_{ij} the following formulae hold

$$\delta_\eta g^{ij} = g^{kj}\nabla_k\delta\eta^i + g^{ik}\nabla_k\delta\eta^j$$
$$\delta_\eta g_{ij} = -g_{kj}\nabla_i\delta\eta^k - g_{ik}\nabla_j\delta\eta^k$$

Since for a transformation of the coordinates (3.1) the vector of covariant differentiation $\mathfrak{s}^i\nabla_i$ is invariant, the expressions for the variations $\delta_\eta\nabla_i$ may also be obtained from (3.5). In particular

$$\delta_\eta\nabla_i T^{j\cdot}_{\cdot k} = -\nabla_s T^{j\cdot}_{\cdot k}\nabla_i\delta\eta^s + \nabla_i T^{s\cdot}_{\cdot k}\nabla_s\delta\eta^j - \nabla_i T^{j\cdot}_{\cdot s}\nabla_k\delta\eta^s$$

For transformation of coordinates (3.1) variation of tensors of types $\mu^{(A)}$ and $\mathbf{K}^{(B)}$ are obtained using formulae that follow from (3.5).

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